Int. J. Solids Structures, 1972, Vol. 8, pp. 1043 to 1050. Pergamon Press. Printed in Great Britain

APPLICATION OF CONTINUUM DISLOCATION THEORY TO GEOMETRY OF LÜDERS FRONT

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Abstract—The geometry of Lüders band fronts was considered from the energy minimum criterion. The elastic energy was calculated from the internal stress field produced by the constraint imposed on the Lüders band by the adjoining regions.

To calculate this internal stress, the continuum theory of dislocation was utilized. The calculation indicated the existence of the particular inclination angle of the stable Lüders fronts, which agreed with the prediction of the theories of Hill and Nadai when the Poisson ratio is zero.

1. INTRODUCTION

THE dislocation theory has been used for the study of the strength of metals and for the microscopic explanation of plastic deformation of crystalline materials. On the other hand, the classical theory of continuum plasticity such as bending, torsion and punching problems has been re-examined on the basis of the knowledge of properties of individual dislocations [1, 2]. In these studies, the sum of dislocation stress and applied stress, both of which are derived from the theory of elasticity, constitutes the stress field in continuum plasticity.

In a simple tension test, plastic deformation usually occurs uniformly throughout a specimen and no dislocation is introduced into the specimen in a statistical sense. However, certain materials exhibit non-uniform deformation in a simple tension test. This non-uniform deformation is characterized by the presence of Lüders bands, which alone undergo plastic deformation. Therefore, dislocations are present at the edge of a Lüders front which separates a plastic region (a Lüders band) from an undeformed region [3, 4]. These dislocations produce an internal stress field. The presence of internal stresses is also known from macroscopic considerations. For there is constraint imposed on the plastic region by the adjoining undeformed region. The presence of internal stresses produces an increase of elastic energy. The internal stress and the elastic energy associated with a Lüders band depend on the geometry of the Lüders front. Thus, when a Lüders band takes a particular geometry, the elastic energy takes a minimum value while other conditions such as plastic strain in the Lüders band are kept constant. This particular geometry corresponds to the stable geometry of a Lüders front according to the energy minimum principle. In the present paper, the internal stress produced by formation of a Lüders band will be calculated with the aid of the dislocation theory and the stable geometry of a Lüders front will be discussed after calculating the elastic energy associated with a Lüders band.

Geometry of Lüders bands has been treated by Hill [5] and Nadai [6]. In these theories, the direction of a Lüders front coincides with that along which the plastic extension vanishes. This situation is considered not to introduce much constraint on the plastic domain from the adjoining undeformed region. However, in these theories, account was not taken of the internal stress. The present study starts from recognition of this internal stress. It will be shown that the indication of the present calculation coincides with the prediction of Hill's and Nadai's theories in a particular case, which reflects the significance of the role played by the mutual constraints between the plastic region and the undeformed region.

2. MODEL AND CALCULATIONS

Let us consider a flat tensile sheet specimen under a uniaxial tensile stress σ^A along the x_3 direction. Suppose a narrow plastic zone (a Lüders band) is formed which inclines at angle θ from the tensile direction, as shown in Fig. 1. For convenience, the width of the plastic zone, w, is assumed to be small compared to the specimen dimensions. The volume constant law requires that

$$\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = 0, \tag{1}$$

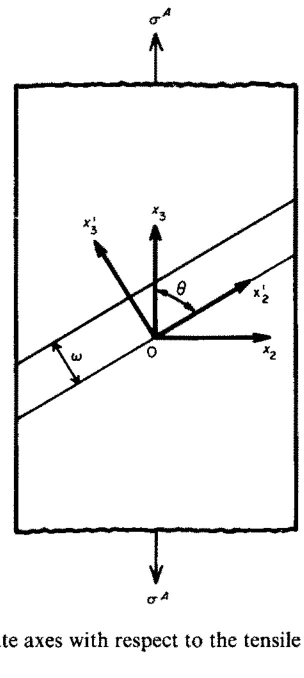

FIG. 1. Definition of coordinate axes with respect to the tensile direction and the plastic zone.

where ε_{11}, ε_{22} and ε_{33} are the diagonal components of the plastic distortion tensor e_{ij} in the plastic zone. Let r be the ratio of lateral plastic contraction along the x_2 direction to the tensile plastic elongation along the x_3 direction. Then, because of equation (1),

$$\varepsilon_{22} = -r\varepsilon_{33} \quad \text{and} \quad \varepsilon_{11} = -(1-r)\varepsilon_{33} \qquad (0 \leq r \leq 1). \tag{2}$$

If plastic deformation in the plastic zone is assumed to occur symmetrically with respect to the sheet plane and the tensile direction, the off-diagonal components of the plastic distortion all vanish. Therefore, the plastic distortion ε_{ij} is given by

$$\varepsilon_{ij} = \begin{pmatrix} -(1-r)\varepsilon_{33} & 0 & 0 \\ 0 & -r\varepsilon_{33} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} \tag{3a}$$

inside the plastic zone and

$$\varepsilon_{ij} = 0 \tag{3b}$$

outside the plastic zone. If the (x'_1, x'_2, x'_3) coordinate system is introduced, as shown in Fig. 1, the plastic distortion ε'_{ij} in this system is expressed as

$$\varepsilon'_{11} = -(1-r)\varepsilon_{33}\{H(x'_3) - H(x'_3 - w)\} \tag{4a}$$

$$\varepsilon'_{22} = (\cos^2\theta - r\sin^2\theta)\varepsilon_{33}\{H(x'_3) - H(x'_3 - w)\} \tag{4b}$$

$$\varepsilon'_{33} = (\sin^2\theta - r\cos^2\theta)\varepsilon_{33}\{H(x'_3) - H(x'_3 - w)\} \tag{4c}$$

$$\varepsilon'_{23} = \varepsilon'_{32} = (1+r)\sin\theta\cos\theta\varepsilon_{33}\{H(x'_3) - H(x'_3 - w)\} \tag{4d}$$

$$\varepsilon'_{12} = \varepsilon'_{21} = \varepsilon'_{13} = \varepsilon'_{31} = 0, \tag{4e}$$

where $H(x)$ is the Heaviside unit step function.

According to the theory of continuum dislocations [3, 4], the dislocation density α'_{hj} is given by

$$\alpha'_{hj} = -\sum_{i=1}^{3}\sum_{l=1}^{3} \varepsilon_{hli}\frac{\partial \varepsilon'_{ij}}{\partial x'_l}. \tag{5}$$

Here α'_{hj} is the x'_j component of the net Burgers vector of the dislocations threading a unit area normal to the x'_h axis, and ε_{hli} is the unit permutation tensor. From equations (4) and (5), we have

$$\alpha'_{21} = -(\partial\varepsilon'_{11}/\partial x'_3) = (1-r)\varepsilon_{33}\{\delta(x'_3) - \delta(x'_3 - w)\} \tag{6a}$$

$$\alpha'_{12} = \partial\varepsilon'_{22}/\partial x'_3 = (\cos^2\theta - r\sin^2\theta)\varepsilon_{33}\{\delta(x'_3) - \delta(x'_3 - w)\} \tag{6b}$$

$$\alpha'_{13} = \partial\varepsilon'_{23}/\partial x'_3 = (1+r)\sin\theta\cos\theta\varepsilon_{33}\{\delta(x'_3) - \delta(x'_3 - w)\}, \tag{6c}$$

where $\delta(x)$ is the Dirac delta function. The other components of α'_{hj} are zero. The geometrical meaning of equation (6a) is that the edge dislocations lying along the x'_2 axis with the infinitesimal Burgers vector parallel to the x'_1 axis are continuously distributed on the boundaries ($x'_3 = 0$ and $x'_3 = w$) between the plastic zone and non-plastic zone. The magnitude of Burgers vector of the infinitesimal edge dislocation lying in the width dX'_1 on the boundary $x'_3 = 0$ is $(1-r)\varepsilon_{33}\,dX'_1$. A similar understanding of geometrical meaning applies to equations (6b) and (6c). These infinitesimal edge dislocations on the boundary $x'_3 = 0$ are schematically shown in Fig. 2.

The internal stress fields due to the above continuous distributions of the infinitesimal dislocations can be constructed from the integration of the stress field of a single dislocation.

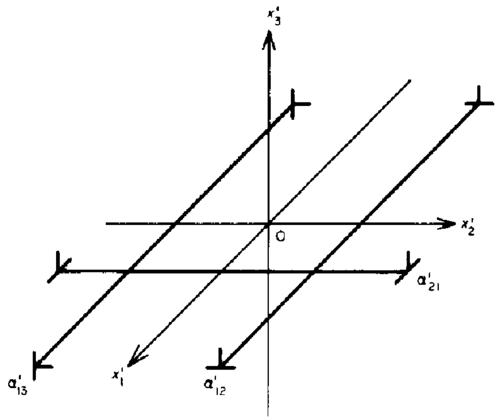

FIG. 2. Schematic presentation of dislocation distributions at the front of a plastic zone.

For example, the stress due to the continuous dislocation α'_{21} on the boundary $x'_3 = 0$ is calculated as

$$\sigma'_{11} = \frac{\mu(1-r)}{2\pi(1-\nu)}\varepsilon_{33}\int_{-\infty}^{\infty}\frac{x'_3\{3(x'_1-X'_1)^2+x'^2_3\}}{\{(x'_1-X'_1)^2+x'^2_3\}^2}\,\mathrm{d}X'_1$$

$$= \begin{cases} \dfrac{\mu(1-r)}{1-\nu}\varepsilon_{33} & (x'_3 > 0) \\ 0 & (x'_3 = 0) \\ -\dfrac{\mu(1-r)}{1-\nu}\varepsilon_{33} & (x'_3 < 0), \end{cases} \tag{7a}$$

$$\sigma'_{22} = \frac{\mu\nu(1-r)}{\pi(1-\nu)}\varepsilon_{33}\int_{-\infty}^{\infty}\frac{x'_3}{(x'_1-X'_1)^2+x'^2_3}\,\mathrm{d}X'_1$$

$$= \begin{cases} \dfrac{\mu\nu(1-r)}{1-\nu}\varepsilon_{33} & (x'_3 > 0) \\ 0 & (x'_3 = 0) \\ -\dfrac{\mu\nu(1-r)}{1-\nu}\varepsilon_{33} & (x'_3 < 0), \end{cases} \tag{7b}$$

$$\sigma'_{33} = -\frac{\mu(1-r)}{2\pi(1-\nu)}\varepsilon_{33}\int_{-\infty}^{\infty}\frac{x'_3\{(x'_1-X'_1)^2-x'^2_3\}}{\{(x'_1-X'_1)^2+x'^2_3\}^2}\,\mathrm{d}X'_1 = 0, \tag{7c}$$

$$\sigma'_{31} = -\frac{\mu(1-r)}{2\pi(1-\nu)}\varepsilon_{33}\int_{-\infty}^{\infty}\frac{(x'_1-X'_1)\{(x'_1-X'_1)^2-x'^2_3\}}{\{(x'_1-X'_1)^2+x'^2_3\}^2}\,\mathrm{d}X'_1 = 0, \tag{7d}$$

and

$$\sigma'_{12} = \sigma'_{23} = 0, \tag{7e}$$

where μ is the shear modulus and ν the Poisson ratio. In a similar manner, the stress due to the presence of continuous dislocations α'_{12} and α'_{13} distributed on the boundary $x'_3 = 0$ can be calculated. Superimposing these stress fields, we obtain the internal stress due to the dislocations α'_{21}, α'_{12} and α'_{13} on the boundary $x'_3 = 0$ as follows:

$$\sigma'_{11} = \begin{cases} \frac{\mu}{1-\nu}\varepsilon_{33}\{(1-r)-\nu(\cos^2\theta - r\sin^2\theta)\} & (x'_3 > 0) \\ 0 & (x'_3 = 0) \\ -\frac{\mu}{1-\nu}\varepsilon_{33}\{(1-r)-\nu(\cos^2\theta - r\sin^2\theta)\} & (x'_3 < 0), \end{cases} \tag{8a}$$

$$\sigma'_{22} = \begin{cases} \frac{\mu}{1-\nu}\varepsilon_{33}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\} & (x'_3 > 0) \\ 0 & (x'_3 = 0) \\ -\frac{\mu}{1-\nu}\varepsilon_{33}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\} & (x'_3 < 0). \end{cases} \tag{8b}$$

The other components of σ'_{ij} are zero.

The total internal stress of the specimen due to the dislocations on the boundaries $x'_3 = 0$ and $x'_3 = w$ is constructed from the results shown in equation (8), since the Burgers vectors of the dislocations on the boundary $x'_3 = w$ are opposite in sign to those of the dislocations on the boundary $x'_3 = 0$. The results are

$$\sigma'_{11} = \frac{2\mu}{1-\nu}\varepsilon_{33}\{(1-r)-\nu(\cos^2\theta - r\sin^2\theta)\} \qquad (0 < x'_3 < w) \tag{9a}$$

and

$$\sigma'_{22} = \frac{2\mu}{1-\nu}\varepsilon_{33}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\} \qquad (0 < x'_3 < w) \tag{9b}$$

and all other components are zero. However, the above solution of equation (9a) is physically unacceptable, since flat sheet surfaces of the specimen are free from tractions. Therefore, a traction which cancels σ'_{11} in equation (9a) is applied. This will cause disturbance in other stress components in the neighborhood of the boundary of the plastic zone. However, for simplicity this disturbance is neglected. The final result of the stress distribution is shown in Fig. 3, where only the non-vanishing component, σ'_{22}, is represented.

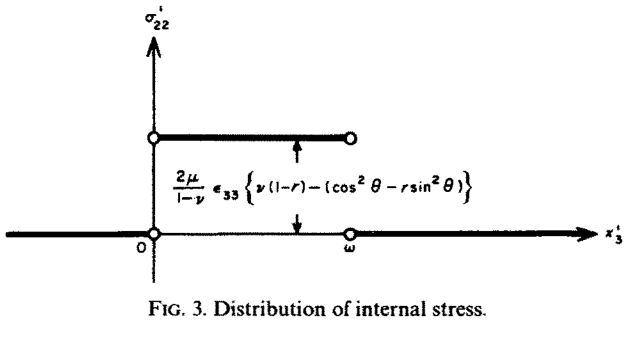

FIG. 3. Distribution of internal stress.

3. ENERGY CONSIDERATION AND INCLINATION ANGLE OF LÜDERS FRONTS

Next, we will find the particular angle of Lüders front inclination at which the specimen is mechanically stable, from the criterion of the energy minimum principle. The energy needed for consideration of mechanical stability is the elastic energy of the specimen plus the change of the external potential. However, the increase of the elastic energy and the change of the external potential related to the elastic deformation of the specimen have nothing to do with the plastic deformation. Therefore, these terms are omitted. Further, once the volume and the plastic strain of the plastic zone are prescribed, the change of the external potential by the plastic deformation is independent of the inclination angle, θ, of the plastic zone. Therefore, the stability of the specimen under the condition of the prescribed volume and plastic strain of the plastic zone can be considered by examining the variation of the elastic energy of the specimen with respect to the inclination angle. By using the internal stress shown in Fig. 3, this elastic energy, E_s, is calculated as

$$E_s = \frac{V\mu\varepsilon_{33}^2}{(1+\nu)(1-\nu)^2}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\}^2, \tag{10}$$

where V is the prescribed volume of the plastic zone. A stable direction of the plastic zone is found under the conditions $\mathrm{d}E_s/\mathrm{d}\theta = 0$ and $\mathrm{d}^2E_s/\mathrm{d}\theta^2 > 0$. This is achieved when θ takes particular values of θ_e given by

$$\sin\theta_e = \pm\sqrt{\frac{1-\nu(1-r)}{1+r}}. \tag{11}$$

It is evident that this energy minimum is attained when the internal stresses all vanish. The angle θ_e is plotted against r for several representative values of the Poisson ratio in Fig. 4.

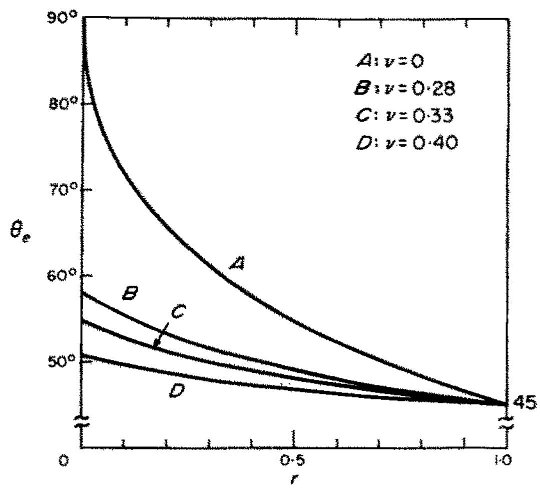

FIG. 4. Relation between the equilibrium angle of a plastic zone and the value, r.

4. DISCUSSION

As shown above, the energy minimum criterion requires the existence of a particular inclination angle of the stable Lüders fronts. This inclination angle depends not only on the ratio r but also on the Poisson ratio of the material. It is seen that when the Poisson ratio is zero, the prediction of the present study, equation (11), agrees with the formula given by Hill [5]. Nadai's postulate [6] that a Lüders front is formed along the zero-extension direction also agrees with equation (11) when the Poisson ratio is zero. The reason for the above agreements is not clear at present. However, it is believed that this indicates the importance of the lateral constraint on a Lüders band imposed by the adjoining undeformed zones.

However, when the Poisson ratio is not equal to zero, the energy minimum is not achieved when Hill and Nadai's criterion is satisfied, and the angle predicted by equation (11) is always smaller than that for the case of $\nu = 0$. As seen in Fig. 4, the effects of the Poisson ratio are prominent where r is close to zero. For example, when $r = 0$, a material with $\nu = 0$ takes $\theta_e = 90°$ while a practical material with ν between 0·25 and 0·35 takes θ_e between 60 and 54°.

Druyvesteyn *et al.* [7] examined the direction of deformation bands in the case where plastic deformation is not symmetric with respect to the tensile direction and in which plastic distortion in the bands is given by

$$\varepsilon_{ij} = \begin{pmatrix} -(1-r)\varepsilon_{33} & 0 & 0 \\ 0 & -r\varepsilon_{33} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

in the present notation. The present model can also deal with the above situation. After performing the calculations and using the arguments in Sections 2 and 3, above, the following results are obtained. The non-vanishing component of the internal stress is σ'_{22} and is given by

$$\sigma'_{22} = \frac{2\mu}{1-\nu}[\varepsilon_{33}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\} - (\varepsilon_{23}+\varepsilon_{32})\sin\theta\cos\theta]$$

within the band. The elastic energy, E_s, is

$$E_s = \frac{V\mu}{(1+\nu)(1-\nu)^2}[\varepsilon_{33}\{\nu(1-r)-(\cos^2\theta - r\sin^2\theta)\} - (\varepsilon_{23}+\varepsilon_{32})\sin\theta\cos\theta]^2.$$

The elastic energy takes a minimum value when σ'_{22} in the deformed band is zero. This is achieved when the inclination angle is given by

$$\tan\theta_e = \frac{(\varepsilon_{32}+\varepsilon_{23}) \pm \sqrt{(\varepsilon_{32}+\varepsilon_{23})^2 + 4\{r+\nu(1-r)\}\{1-\nu(1-r)\}\varepsilon_{33}^2}}{2\{r+\nu(1-r)\}\varepsilon_{33}}. \quad (12)$$

When $\nu = 0$, this result becomes identical to that of Druyvesteyn *et al.* They used Nadai's criterion (zero extension along the band direction) and Hill's theory to calculate the angle θ.

In some metal single crystals, the slip bands are seen to propagate through a tensile specimen as Lüders bands. When there is an active slip system, whose slip plane normal and slip direction are parallel to the specimen surface, the situation considered in the preceding paragraph is directly applied, by taking $r = 1$, $\varepsilon_{33} = (\gamma/2)\sin 2\alpha$ and $(\varepsilon_{23}+\varepsilon_{32}) = -\gamma\cos 2\alpha$. Here, γ is the plastic shear distortion carried by the plastic zone along the slip direction on the slip plane and α is the angle between the tensile direction and the slip direction. Equation (12) predicts the inclination angle of the plastic zone given by

$$\theta = \alpha \quad \text{or} \quad \theta = \alpha + \pi/2.$$

These two conditions are well understood from the dislocation theory. The first condition corresponds to the case where all the dislocations glide out of the specimen. The second condition is equivalent to the configuration in which the edge dislocations advancing at the tip of the glide planes in the Lüders band form a small angle boundary of the tilt type.

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(*Received* 1 *October* 1971)

Абстракт—На основе критерии минимума энергии, исследуется геометрия фронтов полос Лидерса. Упругая энергия подсчитана из поля внутренних напряжений, вызванных связами присоединенных районов, наложенными на полосу Лидерса. Для расчета этих внутренних напряжений, используется сплошная теория дислокации. Расчет указывает наличие частного угла наклона стабильных фронтов лидерса, которое согласовуется с предсказаниями теорий Хилла и Надаи, для случаев, когда коэффициент Пуассона равен нулю.